

GENERAL FRAME STRUCTURES ON QUANTUM PRINCIPAL BUNDLES

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ABSTRACT. A noncommutative-geometric generalization of the classical formalism of frame bundles is developed, incorporating into the theory of quantum principal bundles the concept of the Levi-Civita connection. The construction of a natural differential calculus on quantum principal frame bundles is presented, including the construction of the associated differential calculus on the structure group. General torsion operators are defined and analyzed. Illustrative examples are presented.

1. INTRODUCTION

The formalism of principal bundles plays a central role in the foundation of classical differential geometry. However as far as basic differential-geometric structures are concerned, the appropriate conceptual framework is given by a more restrictive class of frame bundles. Generally speaking, frame bundles are understandable as (covering bundles) of the appropriate subbundles of the principal bundle of linear frames associated to the base manifold. These bundles intrinsically express the geometrical structure existing on the base.

Various fundamental concepts and ideas of theoretical physics are also most naturally and effectively formulated and studied at the language of principal bundles. As a paradigmatic example, let us mention classical and quantum gauge theories. The basic example of applying the formalism of frame bundles in theoretical physics is given by general relativity theory.

In this paper we are going to incorporate the classical theory of principal frame bundles into the framework of non-commutative differential geometry [1]. All considerations are logically based on a general theory of quantum principal bundles [2, 3] in which the bundle and the base are quantum objects, and quantum groups play the role of the structure groups. We shall consider here only bundles with compact structure groups. Our constructions are motivated by properties of classical frame structures with orthonormal frames, which preassumes the existence of the metrics on the base manifold. However, the developed formalism can be applied in the general context, dealing with bundles with arbitrary structure groups.

The paper is organized in the following way. The next section is devoted to the definition of frame structures on quantum principal bundles. In classical geometry, every framed bundle involving orthonormal frames canonically determines the unique metric connection with vanishing torsion. This is by definition the Levi-Civita connection. It is also of the special interest to consider subbundles of the orthonormal frame bundle, invariant under the Levi-Civita connection.

The starting idea of the next section is to generalize the classical concept of coordinate first-order forms [11] which determine the association of orthonormal frames of the tangent spaces of the base manifold, to the points of the corresponding fibers

of the bundle. This line of thinking was also suggested in [10], in a conceptually different context. Formally, we shall start from a vector space \mathbb{V} representing abstract quantum first-order coordinate horizontal forms, which is equipped with the appropriate additional structure, equivalent to specifying a bicovariant $*$ -bimodule Ψ over [13] the structure quantum group G . The space \mathbb{V} is then re-interpretable as the left-invariant part of Ψ . In particular, there exists the intrinsic braid operator $\tau: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$. Using this operator, we can construct the counterpart of the corresponding exterior algebra \mathbb{V}^\wedge , requiring the τ -antimultiplicativity between the elements of \mathbb{V} . In our context this corresponds to the subalgebra of horizontal forms generated by coordinate forms. Finally, the complete algebra \mathfrak{hor}_P representing horizontal forms can be constructed by taking the natural cross product between \mathbb{V}^\wedge and the algebra \mathcal{B} representing the (appropriate functions on the) quantum principal bundle P . The concept of a frame structure is completed by introducing the analogs of Levi-Civita connections. In the formalism, they will be represented by certain antiderivations $\nabla: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$, playing the role of the covariant derivative maps of actual Levi-Civita connections.

Starting from a framed quantum principal bundle P , and applying a general construction presented in [4], it is possible to construct in a natural manner the whole differential calculus $\Omega(P)$ on P , including a natural bicovariant first-order $*$ -calculus Γ on G . This problematics is discussed in Section 3. In particular, it turns out that \mathfrak{hor}_P coincides with the algebra of horizontal forms associated to $\Omega(P)$, as defined in the framework of the general theory [3]. Moreover, regular (and multiplicative) connections [3] on P are in a natural correspondence with special antiderivations $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$. These maps play the role of the corresponding covariant derivative maps. In particular, the construction justifies the mentioned interpretation of the frame structure.

In Section 4 we shall introduce and analyze counterparts of torsion operators, associated to arbitrary connections on P . Section 5 gives the explicit ‘coordinate’ description of covariant derivative maps D , in terms of the analogs of associated coordinate horizontal vector fields.

Finally, in Section 6 some interesting examples are collected, and some concluding remarks are made. A large class of examples of quantum frame bundles comes from the appropriate quantum homogeneous spaces [3] and bundles associated to them. It turns out that if G is a subgroup of a quantum group H , and if H is equipped with a special first-order differential calculus, then every principal H -bundle becomes a framed principal G -bundle, after restricting the action map to the subgroup G . A theory of quantum frame bundles with classical structure groups [6] will be also considered from the point of view of the general formalism. A particular attention will be given to the framed quantum line bundles, where it is possible to write down simple expressions for all basic entities figuring in the game.

2. FRAMED QUANTUM PRINCIPAL BUNDLES

Let G be a compact matrix quantum group [13], represented by a Hopf $*$ -algebra \mathcal{A} . The elements of \mathcal{A} are interpreted as ‘polynomial functions’ on G . The coproduct, counit and the antipode map will be denoted by ϕ , ϵ and κ respectively.

Let us consider a bicovariant [13] bimodule Ψ over G , equipped with the left/right action maps $\ell_\Psi: \Psi \rightarrow \mathcal{A} \otimes \Psi$ and $\rho_\Psi: \Psi \rightarrow \Psi \otimes \mathcal{A}$. There exists a natural decomposition $\Psi \leftrightarrow \mathcal{A} \otimes \mathbb{V}$, where $\mathbb{V} = \Psi_{inv}$ is the corresponding left-invariant part.

The bicovariant bimodule structure on Ψ is re-expressed via the right action $\varkappa = (\wp_\Psi \upharpoonright \mathbb{V}): \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$ and the natural right \mathcal{A} -module structure $\circ: \mathbb{V} \otimes \mathcal{A} \rightarrow \mathbb{V}$, where $\vartheta \circ a = \kappa(a^{(1)})\vartheta a^{(2)}$.

Let us assume that Ψ is also $*$ -covariant. Then the space \mathbb{V} is $*$ -invariant and the following compatibility conditions hold:

$$\begin{aligned} \varkappa* &= (* \otimes *)\varkappa & (\vartheta \circ a)^* &= \vartheta^* \circ \kappa(a)^* \\ \varkappa(\vartheta \circ a) &= \sum_k (\vartheta_k \circ a^{(2)}) \otimes \kappa(a^{(1)})c_k a^{(3)}, \end{aligned}$$

where $\sum_k \vartheta_k \otimes c_k = \varkappa(\vartheta)$.

Let us denote by $\tau: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$ the canonical braid operator [13] associated to Ψ . It is expressed via \varkappa and \circ as follows

$$\tau(\eta \otimes \vartheta) = \sum_k \vartheta_k \otimes (\eta \circ c_k).$$

Let \mathbb{V}^\wedge be the corresponding τ -exterior algebra, obtained by factorizing the tensor algebra \mathbb{V}^\otimes through the quadratic relations $\text{im}(I + \tau)$. At this point it is reasonable to assume that the braid operator τ is such that $\ker(I + \tau) \neq \{0\}$. This condition ensures the nontriviality of the higher-order part of \mathbb{V}^\wedge .

Let us also assume that the space \mathbb{V} is equipped with a scalar product, such that the action $\varkappa: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$ is unitary. Let us fix an orthonormal basis $\{\theta_1, \dots, \theta_n\}$ in the space \mathbb{V} . In this basis, the action is described by a unitary matrix $\{u_{ij}\}$, given by $\varkappa(\theta_i) = \sum_j \theta_j \otimes u_{ji}$.

The $*$ -structure, \varkappa and \circ are naturally extendible to $\mathbb{V}^{\otimes, \wedge}$, by requiring

$$\begin{aligned} (\eta\vartheta) \circ a &= (\eta \circ a^{(1)})(\vartheta \circ a^{(2)}) & 1 \circ a &= \epsilon(a)1 \\ (\eta\vartheta)^* &= (-1)^{\partial\eta\partial\vartheta}\vartheta^*\eta^* & \varkappa(\eta\vartheta) &= \varkappa(\eta)\varkappa(\vartheta). \end{aligned}$$

The extended maps define bicovariant graded $*$ -algebras $\Psi^{\otimes, \wedge} \leftrightarrow \mathcal{A} \otimes \mathbb{V}^{\otimes, \wedge}$.

Let $P = (\mathcal{B}, i, F)$ be a quantum principal G -bundle [3] over a quantum space M . By definition, \mathcal{B} is a $*$ -algebra representing P at the level of quantum spaces, while $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ is a $*$ -homomorphism playing the role of the dualized right action of G on P . Finally, $i: \mathcal{V} \rightarrow \mathcal{B}$ is the dualized projection of P on M , and \mathcal{V} is the $*$ -algebra representing M . The image of i coincides with the F -fixed-point subalgebra of \mathcal{B} . Geometrically this means that M is interpretable as the orbit space associated to P , relative to the action of the structure group. We shall identify the elements of \mathcal{V} with their images in \mathcal{B} .

Lemma 1. (i) *The formulas*

$$\begin{aligned} (q \otimes \vartheta)(b \otimes \eta) &= \sum_k qb_k \otimes (\vartheta \circ c_k)\eta \\ (b \otimes \vartheta)^* &= \sum_k b_k^* \otimes (\vartheta^* \circ c_k^*) \end{aligned}$$

where $\sum_i b_i \otimes c_i = F(b)$, determine a $*$ -algebra structure on the graded vector space $\mathfrak{hor}_P = \mathcal{B} \otimes \mathbb{V}^\wedge$. In particular, $\mathcal{B} = \mathfrak{hor}_P^0$ and \mathbb{V}^\wedge is a graded $*$ -subalgebra of $\mathfrak{hor}(P)$.

(ii) *There exists the unique $*$ -homomorphism $F^\wedge: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P \otimes \mathcal{A}$ extending actions F and $\varkappa: \mathbb{V}^\wedge \rightarrow \mathbb{V}^\wedge \otimes \mathcal{A}$. The space \mathfrak{hor}_P is a graded \mathcal{A} -comodule.* \square

The above definition of the product in \mathfrak{hor}_P , together with the definition of \mathbb{V}^\wedge , implies commutation relations

$$\vartheta\varphi = (-1)^{\partial\varphi\partial\vartheta}\varphi_k(\vartheta \circ c_k),$$

where $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$, while $\varphi \in \mathfrak{hor}_P$ and $\vartheta \in \mathbb{V}^\wedge$.

We pass to the definition of frame structures.

Definition 1. A frame structure on a quantum principal bundle P relative to a bicovariant $*$ -bimodule Ψ is a first-order hermitian antiderivation $\nabla: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ such that

(i) The following equalities hold

$$(2.1) \quad F^\wedge \nabla = (\nabla \otimes \text{id}) F^\wedge$$

$$(2.2) \quad \nabla(\mathcal{H}) = \{0\},$$

where $\mathcal{H} \subseteq \mathfrak{hor}_P$ is the $*$ -subalgebra generated by $\nabla(\mathcal{V})$ and \mathbb{V} .

(ii) There exist elements $b_{\alpha i} \in \mathcal{B}$ and $f_\alpha \in \mathcal{V}$ such that

$$(2.3) \quad 1 \otimes \theta_i = \sum_\alpha b_{\alpha i} \nabla(f_\alpha)$$

$$(2.4) \quad F(b_{\alpha i}) = \sum_j b_{\alpha j} \otimes u_{ji},$$

for each $i \in \{1, \dots, n\}$.

Geometrically, the definition of a framed bundle incorporates the idea that every point of the bundle gives rise to an orthonormal system in the corresponding tangent space, spanning a complement to the vertical subspace. These complements form a special connection on the bundle (actually the Levi-Civita connection), and the map ∇ is the corresponding covariant derivative. An alternative possible geometrical interpretation of the frame structures is that they represent the appropriate ‘metrics’ on the base space M .

3. DIFFERENTIAL CALCULUS ON FRAMED BUNDLES

In this section we describe the construction of the intrinsic differential calculus, which can be associated to every quantum principal bundle P endowed with a frame structure. The construction gives also a natural differential calculus on the structure group G .

Let $\Omega_M \subseteq \mathfrak{hor}_P$ be the F^\wedge -fixed point subalgebra. This is a graded $*$ -subalgebra of \mathfrak{hor}_P and we have $\Omega_M^0 = \mathcal{V}$.

Let us fix a frame structure $\nabla: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$. The F^\wedge -covariance of ∇ implies that Ω_M is ∇ -invariant. In what follows, we shall denote by $\overset{M}{d}: \Omega_M \rightarrow \Omega_M$ the corresponding restriction map.

Lemma 2. (i) The algebra Ω_M is generated by elements of the form $\{f, \overset{M}{d}(f)\}$, where $f \in \mathcal{V}$.

(ii) The map $\overset{M}{d}$ is a hermitian differential on Ω_M .

Proof. Inductively applying equation (2.3) and the basic commutation relation, it follows that the elements from \mathfrak{hor}_P^m are of the form

$$\varphi = \sum b \prod_{i=1}^m \overset{M}{d}(f_i),$$

where $b \in \mathcal{B}$ and $f_i \in \mathcal{V}$. If $\varphi \in \Omega_M$ then it is possible to assume that $b \in \mathcal{V}$, as follows from the fact that there exists a natural projection map $p_0: \mathfrak{hor}_P \rightarrow \Omega_M$ given by

$$p_0 = (\text{id} \otimes h)F^\wedge,$$

where $h: \mathcal{A} \rightarrow \mathbb{C}$ is the Haar measure [12] on G . This proves (i).

To complete the proof, it is sufficient to apply (i) and to observe that the square of ${}^M\mathcal{d}$ vanishes on \mathcal{V} . It follows that ${}^M\mathcal{d}^2 = 0$, globally. \square

Let us consider a real affine space $\mathfrak{der}(P)$ consisting of hermitian first-order antiderivations $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ intertwining the map F^\wedge and extending the differential ${}^M\mathcal{d}$. By definition, $\nabla \in \mathfrak{der}(P)$. Applying the results of [4], it follows that for each $D \in \mathfrak{der}(P)$ there exist the unique linear map $\varrho_D: \mathcal{A} \rightarrow \mathfrak{hor}_P$ such that

$$(3.1) \quad D^2(\varphi) = - \sum_k \varphi_k \varrho_D(c_k),$$

where $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$.

Definition 2. The map ϱ_D is called the curvature of D .

As explained in [4], the curvature ϱ_D also satisfies the following identities:

$$\begin{aligned} F^\wedge \varrho_D &= (\varrho_D \otimes \text{id})\text{ad} \\ D\varrho_D &= 0 \quad \varrho_D(a)^* = -\varrho_D[\kappa(a)^*] \\ \varrho_D(a)\varphi &= \sum_k \varphi_k \varrho_D(ac_k) + \epsilon(a)D^2(\varphi), \end{aligned}$$

for each $\varphi \in \mathfrak{hor}_P$ and $a \in \mathcal{A}$. Here $\text{ad}: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is the adjoint action of G on itself, given by $\text{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)})a^{(3)}$.

Furthermore, let us consider the real vector space $\overrightarrow{\mathfrak{der}}(P)$ associated to $\mathfrak{der}(P)$. The elements of $\overrightarrow{\mathfrak{der}}(P)$ are interpretable as first-order F^\wedge -covariant hermitian antiderivations on \mathfrak{hor}_P which vanish on Ω_M . Every such a map E can be uniquely represented in the form

$$(3.2) \quad E(\varphi) = -(-1)^{\partial\varphi} \sum_k \varphi_k \chi(c_k)$$

where $\chi = \chi_E: \mathcal{A} \rightarrow \mathfrak{hor}_P$ is a first-order linear map. According to [4], the following equalities hold

$$(3.3) \quad F^\wedge \chi(a) = (\chi \otimes \text{id})\text{ad}(a)$$

$$(3.4) \quad \chi[\kappa(a)^*] = -\chi(a)^*$$

$$(3.5) \quad \chi(a)\varphi = \epsilon(a)E(\varphi) + (-1)^{\partial\varphi} \sum_k \varphi_k \chi(ac_k)$$

for each $E \in \overrightarrow{\mathfrak{der}}(P)$. Conversely, every first-order linear map χ satisfying the above equalities determines an antiderivation E , via equality (3.2).

Lemma 3. *Every linear map $\chi = \chi_E$ satisfying the above mentioned properties is completely determined by its values on the matrix elements of an arbitrary faithful representation of G .*

Proof. Let $T \in M_k(\mathbb{C})$ be a faithful unitary representation of G . This means that the matrix elements of T generate the $*$ -algebra \mathcal{A} . According to (3.5) we have

$$(3.6) \quad \chi(aT_{ij}) = \sum_{\alpha} q_{\alpha i} \chi(a) p_{\alpha j},$$

where $a \in \ker(\epsilon)$ and the elements $q_{\alpha i}, p_{\alpha i} \in \mathcal{B}$ satisfy

$$F(p_{\alpha i}) = \sum_j p_{\alpha j} \otimes T_{ji} \quad \sum_{\alpha} q_{\alpha i} p_{\alpha j} = \delta_{ij} 1.$$

We can assume without a lack of generality that T is self-conjugate. Inductively applying (3.6) it follows that the values of χ are algebraically expressible in terms of its values on matrix elements T_{ij} . \square

Equation (3.6) implies in particular the following generalization of the corresponding classical antisymmetry property

$$(3.7) \quad \chi(T_{ij}) + \sum_{k\alpha} q_{\alpha k} \chi(T_{ki}^*) p_{\alpha j} = 0.$$

It is worth noticing that the same reasoning applies to the curvature maps ϱ_D , and that the following connecting identity [4] holds

$$(3.8) \quad \varrho_{D+E}(a) = \varrho_D(a) + D\chi_E(a) + \chi_E(a^{(1)})\chi_E(a^{(2)}).$$

The system of maps ϱ_D and χ_E intrinsically determines a bicovariant first-order $*$ -calculus Γ on G . By definition, this calculus is based on the right \mathcal{A} -ideal $\mathcal{R} \subseteq \ker(\epsilon)$ consisting of all elements annihilated by maps ϱ_D and χ_E . As for all left-covariant differential structures, the left-invariant part of Γ is given by $\Gamma_{inv} = \ker(\epsilon)/\mathcal{R}$.

By definition, the maps ϱ_D and χ_E are factorizable through \mathcal{R} . In what follows we shall denote by the same symbols the factorized maps $\varrho_D, \chi_E: \Gamma_{inv} \rightarrow \mathfrak{hor}_P$. The following commutation relations hold

$$(3.9) \quad \chi_E(\vartheta)\varphi = (-1)^{\partial\varphi} \sum_k \varphi_k \chi_E(\vartheta \circ c_k)$$

$$(3.10) \quad \varrho_D(\vartheta)\varphi = \sum_k \varphi_k \varrho_D(\vartheta \circ c_k)$$

for each $\vartheta \in \Gamma_{inv}$ and $\varphi \in \mathfrak{hor}_P$, where \circ is the corresponding right \mathcal{A} -module structure on Γ_{inv} . Furthermore, the maps χ_E and ϱ_D are hermitian, and intertwine ϖ and F^\wedge . Here $\varpi: \Gamma_{inv} \rightarrow \Gamma_{inv} \otimes \mathcal{A}$ is the induced adjoint action, explicitly given by

$$\varpi\pi = (\pi \otimes \text{id})\text{ad},$$

where $\pi: \mathcal{A} \rightarrow \Gamma_{inv}$ is the canonical projection map (playing the role of the germ map).

The described construction gives the minimal calculus Γ , compatible with the whole space $\mathfrak{der}(P)$. The same construction can be performed starting from an arbitrary affine subspace $\mathcal{D} \subseteq \mathfrak{der}(P)$, containing the frame structure ∇ . The minimal choice for \mathcal{D} is to contain only ∇ . In this case the calculus is determined only by the curvature map ϱ_∇ . Interestingly, even in this case the compatibility requirement may imply very strong restrictions on the calculus. It is also worth noticing that generally the compatibility between the appropriately chosen subspace \mathcal{D} will ensure the full compatibility between Γ and $\mathfrak{der}(P)$.

Applying to $\{\mathfrak{hor}_P, F^\wedge, \mathfrak{der}(P), \Omega_M\}$ a general construction presented in [4] we obtain in the intrinsic way a graded-differential $*$ -algebra $\Omega(P)$ representing the complete differential calculus on P , such that \mathfrak{hor}_P is recovered as the corresponding $*$ -subalgebra of horizontal forms for $\Omega(P)$. Moreover, to every map D it is possible to associate intrinsically a connection ω_D on P , such that D is reinterpreted as the covariant derivative of the connection ω_D .

Let us sketch this construction of the calculus $\Omega(P)$. At the level of graded vector spaces we define

$$\Omega(P) = \mathfrak{hor}_P \otimes \Gamma_{inv}^\wedge,$$

while the $*$ -algebra structure is specified by the formulas

$$(3.11) \quad (\psi \otimes \vartheta)(\varphi \otimes \eta) = (-1)^{\partial\varphi\partial\vartheta} \sum_k \psi \varphi_k \otimes (\vartheta \circ c_k) \eta$$

$$(3.12) \quad (\varphi \otimes \vartheta)^* = \sum_k \varphi_k^* \otimes (\vartheta^* \circ c_k^*),$$

where $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$. We have assumed that the complete calculus on the structure group G is based on the universal differential [2] envelope Γ^\wedge of Γ . Another intrinsic choice for the higher-order calculus is the braided exterior [13] algebra Γ^\vee , associated to the natural braiding $\sigma: \Gamma_{inv}^{\otimes 2} \rightarrow \Gamma_{inv}^{\otimes 2}$.

The action of the differential $d: \Omega(P) \rightarrow \Omega(P)$ is given by

$$(3.13) \quad d(\varphi) = \nabla(\varphi) + (-1)^{\partial\varphi} \sum_k \varphi_k \otimes \pi(c_k) \quad d(\vartheta) = d^\wedge(\vartheta) + \varrho_\nabla(\vartheta),$$

and extended to the whole $\Omega(P)$ by requiring the graded Leibniz rule. In the above formulas $\varphi \in \mathfrak{hor}_P$ and $\vartheta \in \Gamma_{inv}^\wedge$, while $d^\wedge: \Gamma_{inv}^\wedge \rightarrow \Gamma_{inv}^\wedge$ is the corresponding differential. We have $d^2 = 0$ and $d* = *d$. Furthermore, the formulas

$$(3.14) \quad \widehat{F}(\varphi) = F^\wedge(\varphi) \quad \widehat{F}(\vartheta) = \varpi(\vartheta) + 1 \otimes \vartheta$$

consistently and uniquely determine a homomorphism $\widehat{F}: \Omega(P) \rightarrow \Omega(P) \widehat{\otimes} \Gamma^\wedge$. It turns out that this map is hermitian, and commutes with the corresponding differentials. Therefore we can say [3] that $\Omega(P)$ represents a differential calculus on a quantum principal bundle P . The algebra \mathfrak{hor}_P is reconstructed as the horizontal part of $\Omega(P)$, in other words

$$\mathfrak{hor}_P = \widehat{F}^{-1}(\Omega(P) \otimes \mathcal{A}).$$

Let us consider an arbitrary $D \in \mathfrak{der}(D)$ and let us define $E = D - \nabla$. The formula

$$(3.15) \quad \omega_D(\vartheta) = 1 \otimes \vartheta + \chi_E(\vartheta)$$

determines a special connection $\omega_D: \Gamma_{inv} \rightarrow \Omega(P)$ on P . This connection is regular and multiplicative [3]. Moreover, the corresponding operators of covariant derivative and curvature coincide with D and ϱ_D respectively.

4. TORSION OPERATORS

In this section we shall introduce general torsion operators, and analyze their algebraic properties. We shall also discuss the question of the uniqueness of the Levi-Civita connection.

Let us start from a framed principal bundle P , and assume that the algebra \mathfrak{hor}_P is included in the complete calculus $\Omega(P)$, as described in the previous section.

Let $\omega: \Gamma_{inv} \rightarrow \Omega(P)$ be an arbitrary connection [3] on P . By definition, this means that ω is a first-order hermitian linear map satisfying

$$\widehat{F}[\omega(\vartheta)] = \sum_k \omega(\vartheta_k) \otimes c_k + 1 \otimes \vartheta,$$

where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$.

Furthermore, the covariant derivative $D_\omega: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ of the connection ω is given by equality

$$D_\omega(\varphi) = d\varphi - \sum_k \varphi_k \omega \pi(c_k),$$

where $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$. As we have already mentioned, particularly interesting are *regular connections*. They are characterized by the fact that D_ω satisfies the graded Leibniz rule.

If the calculus Γ is maximally compatible with the structure of \mathfrak{hor}_P (as the calculus described in the previous section), then we have the following correspondence,

$$\left\{ \text{Regular connections on } P \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Hermitian first-order covariant} \\ \text{antiderivations on } \mathfrak{hor}_P \\ \text{extending } {}^M d: \Omega_M \rightarrow \Omega_M \end{array} \right\} = \mathfrak{der}(P)$$

induced by the covariant derivative map.

The curvature of an arbitrary connection ω is defined by

$$R_\omega = d\omega - \langle \omega, \omega \rangle.$$

This is the analog of the classical first structure equation. Here $\langle \rangle$ are the brackets associated [3] to an arbitrary embedded differential map $\delta: \Gamma_{inv} \rightarrow \Gamma_{inv} \otimes \Gamma_{inv}$. Explicitly, δ is a hermitian map intertwining the corresponding adjoint actions of G and satisfying

$$\delta(\vartheta) = \sum_k \vartheta_k^1 \otimes \vartheta_k^2, \quad d^\wedge(\vartheta) = \sum_k \vartheta_k^1 \vartheta_k^2.$$

Definition 3. A linear map $T_\omega: \mathbb{V} \rightarrow \mathfrak{hor}_P$ defined by

$$(4.1) \quad T_\omega^i = T_\omega(\theta_i) = D_\omega(1 \otimes \theta_i)$$

is called *the torsion* of ω .

Let us also define components of the curvature $R_\omega^{ij} = R_\omega \pi(u_{ij})$. In the following proposition we have collected the most important properties of the torsion operators.

Proposition 4. (i) *The map T_ω is hermitian and the diagram*

$$(4.2) \quad \begin{array}{ccc} \mathbb{V} & \xrightarrow{T_\omega} & \mathfrak{hor}_P \\ \wr \downarrow & & \downarrow F^\wedge \\ \mathbb{V} \otimes \mathcal{A} & \xrightarrow{T_\omega \otimes \text{id}} & \mathfrak{hor}_P \otimes \mathcal{A} \end{array}$$

is commutative.

(ii) We have

$$(4.3) \quad D_\omega T_\omega^i = - \sum_j \theta_j R_\omega^{ji} - \sum_j T_\omega^j \rho_\omega(u_{ji}),$$

where $\rho_\omega: \mathcal{A} \rightarrow \mathfrak{hor}_P^2$ is a linear map given by

$$(4.4) \quad \rho_\omega(a) = \langle \omega, \omega \rangle \pi(a) + \omega \pi(a^{(1)}) \omega \pi(a^{(2)}).$$

The map ρ_ω is related to the lack of the multiplicativity of ω , and in particular for regular connections it vanishes identically.

Proof. Equality (4.3) is a direct consequence of a general expression [3] for the square of the covariant derivative. The diagram (4.2) and the hermicity of T_ω follow from the covariance and hermicity of D_ω . \square

It is worth noticing that (4.3) generalizes the classical second structure equation. According to our definition of frame structures, the torsion of the connection $\omega = \omega_\nabla$ vanishes.

In classical geometry, the torsion tensor (together with the metricity condition) uniquely characterizes every connection. In particular, the Levi-Civita connection is uniquely characterized by the vanishing torsion condition.

In our general framework such a characterization does not longer hold. Let us analyze regular connections having the same torsion. Such connections are grouped into real affine subspaces of $\mathfrak{der}(P)$, which are the orbits under the natural action of the real vector subspace $\mathcal{X} \subseteq \overrightarrow{\mathfrak{der}}(P)$ consisting of linear maps $\chi: \mathcal{A} \rightarrow \mathfrak{hor}_P$ satisfying the additional condition

$$(4.5) \quad \sum_j \theta_j \chi(u_{ji}) = 0.$$

The above condition is a variant of braided-symmetricity of two first indexes in the coefficients χ_{kji} given by $\sum_k \theta_k \chi_{kji} = \chi(u_{ji})$. If the braiding τ is sufficiently ‘regular’ then this symmetricity condition together with the ‘antisymmetricity’ condition (3.7) will imply that $\chi(u_{ij}) = 0$. According to Lemma 3, this means that χ vanishes identically.

Definition 4. We say that a frame structure on P is *regular* iff $\mathcal{X} = \{0\}$. In this case regular connections are completely determined by their torsions, and in particular ∇ is the unique regular connection with the vanishing torsion.

If we pass to a more general framework allowing non-compact structure groups and resigning from the unitarity assumption for $\varkappa: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$ then, generally, the space \mathcal{X} will be ‘large’. This is in a complete agreement with classical geometry.

5. EXPLICIT COORDINATE EXPRESSIONS

Let us first observe that the following natural decomposition holds

$$(5.1) \quad \mathfrak{hor}_P \leftrightarrow \mathcal{B} \otimes_{\mathbb{V}} \Omega_M \leftrightarrow \Omega_M \otimes_{\mathbb{V}} \mathcal{B},$$

induced by the product map. It follows that the action of every covariant derivative map $D \in \mathfrak{der}(P)$ is completely determined by the restriction on \mathcal{B} . In particular, the action on the standard coordinate forms is given by

$$(5.2) \quad D(\theta_i) = \sum_{\alpha} D(b_{\alpha i})^M d(f_{\alpha}).$$

Therefore we can introduce the ‘coordinate’ description of the covariant derivatives, by the formula

$$(5.3) \quad D(b) = \sum_i X_i(b) \otimes \theta_i,$$

where the maps $X_i: \mathcal{B} \rightarrow \mathcal{B}$ are counterparts of horizontal coordinate vectors fields. They completely determine the map D . A particular role is given to the vector fields ∂_i corresponding to ∇ . We have

$$(5.4) \quad \nabla(b \otimes \vartheta) = \sum_i \partial_i(b) \otimes \theta_i \vartheta.$$

We are going to analyze how elementary properties of D are reflected on the maps X_i . At first, D acts on \mathcal{B} as a derivation. This implies

$$(5.5) \quad X_i(qb) = qX_i(b) + \sum_{jk} \mu_{ji}(c_k) X_j(q) b_k,$$

where $\sum_k b_k \otimes c_k = F(b)$ and maps $\mu_{ji}: \mathcal{A} \rightarrow \mathbb{C}$ are given by

$$(5.6) \quad \sum_i \mu_{ji}(a) \theta_i = \theta_j \circ a.$$

The fact that every covariant derivative D extends the differential $\overset{M}{d}: \Omega_M \rightarrow \Omega_M$ implies

$$(5.7) \quad (X_i - \partial_i)(\mathcal{V}) = \{0\}.$$

Further, the covariance of D is equivalent to equalities

$$(5.8) \quad FX_i(b) = \sum_{jk} X_j(b_k) \otimes c_k \kappa^{-1}(u_{ij}).$$

Conversely, let us assume that a system of maps $X_i: \mathcal{B} \rightarrow \mathcal{B}$ is given, satisfying all the above equations. Then the formula

$$(5.9) \quad D(bw) = \sum_i (X_i(b) \otimes \theta_i) w + b \overset{M}{d}(w)$$

consistently defines a first-order linear map $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$. This map intertwines F^\wedge , extends $\overset{M}{d}$ and satisfies the ‘partial’ Leibniz rules

$$\begin{aligned} D(\varphi w) &= D(\varphi)w + (-1)^{\partial \varphi} \varphi \overset{M}{d}(w) \\ D(b\varphi) &= D(b)\varphi + bD(\varphi), \end{aligned}$$

for each $\varphi \in \mathfrak{hor}_P$.

Lemma 5. *The following conditions are equivalent:*

- (i) *The map D satisfies the graded Leibniz rule.*
- (ii) *We have*

$$\overset{M}{d}(f)D(b) + D[\overset{M}{d}(f)b] = 0$$

for each $f \in \mathcal{V}$ and $b \in \mathcal{B}$.

(iii) *It is possible to introduce the curvature map $\varrho_D: \mathcal{A} \rightarrow \mathfrak{hor}_P$, naturally associated to D .* □

Let us now assume that the conditions figuring in the above proposition are satisfied for D . Then the conjugate map D^* is also a covariant first-order antiderivation extending $\overset{M}{d}$, and we have the unique decomposition

$$(5.10) \quad D = D_1 + iD_2, \quad D_1 = \frac{D + D^*}{2} \quad D_2 = \frac{D - D^*}{2i}.$$

with $D_1, D_2 \in \mathfrak{der}(P)$.

6. CONCLUDING EXAMPLES & REMARKS

6.1. Quantum Homogeneous Spaces

A large class of examples of framed quantum principal bundles is associated to the appropriate quantum homogeneous spaces.

Let us consider a quantum group H , represented by a Hopf $*$ -algebra \mathcal{A}' . Let us assume that G is realized as a subgroup of H . This means that a projection $*$ -homomorphism $\varsigma: \mathcal{A}' \rightarrow \mathcal{A}$ is given such that

$$\phi\varsigma = (\varsigma \otimes \varsigma)\phi' \quad \epsilon\varsigma = \epsilon' \quad \kappa\varsigma = \varsigma\kappa'.$$

All primed entities appearing in this subsection refer to the group H .

Let $\mathcal{K} \subseteq \mathcal{A}'$ be the fixed-point subalgebra relative to the natural right action $\phi_*: \mathcal{A}' \rightarrow \mathcal{A}' \otimes \mathcal{A}$ of G . Furthermore, let us denote by $\text{ad}_*: \mathcal{A}' \rightarrow \mathcal{A}' \otimes \mathcal{A}$ the restricted adjoint action, given by $\text{ad}_*(q) = q^{(2)} \otimes \varsigma[\kappa'(q^{(1)})q^{(3)}]$. We have

$$\begin{aligned} \phi'(\mathcal{K}) &\subseteq \mathcal{A}' \otimes \mathcal{K} \\ \text{ad}_*(\mathcal{K}) &\subseteq \mathcal{K} \otimes \mathcal{A}. \end{aligned}$$

Let us consider a first-order $*$ -calculus Φ over H , which is left-covariant and right G -covariant. The last condition means that we can consistently introduce the right action map $\varkappa: \Phi_{\text{inv}} \rightarrow \Phi_{\text{inv}} \otimes \mathcal{A}$, which is given by the formula

$$(6.1) \quad \varkappa\pi' = (\pi' \otimes \text{id})\text{ad}_*.$$

It is worth noticing that if Φ is bicovariant then it is automatically right G -covariant and

$$\varkappa = (\text{id} \otimes \varsigma)\varpi', \quad \varpi': \Phi_{\text{inv}} \rightarrow \Phi_{\text{inv}} \otimes \mathcal{A}'.$$

The calculus Φ is projected via the map ς , onto the bicovariant $*$ -calculus Γ over the group G . Let us denote by the same symbol $\varsigma: \Phi_{\text{inv}} \rightarrow \Gamma_{\text{inv}}$ the corresponding induced map, so that we have $\varsigma(\Phi_{\text{inv}}) = \Gamma_{\text{inv}}$, and

$$(6.2) \quad \varpi\varsigma = (\varsigma \otimes \text{id})\varkappa \quad \varsigma(\vartheta \circ q) = \varsigma(\vartheta) \circ \varsigma(q).$$

It follows that $\mathcal{L} = \ker(\varsigma|_{\Phi_{\text{inv}}})$ is a $*$ -submodule of Φ_{inv} . Furthermore, we have $\pi'(\mathcal{K}) \subseteq \mathcal{L}$. Let us assume that $\pi': \mathcal{K} \rightarrow \mathcal{L}$ is surjective. The space \mathcal{L} is \varkappa -invariant and we have

$$(6.3) \quad \varkappa\pi'(q) = \pi'(q^{(2)}) \otimes \kappa\varsigma(q^{(1)}), \quad \forall q \in \mathcal{K}.$$

Our next assumption is that the right \mathcal{A}' -module structure on Φ_{inv} is projectable to the right \mathcal{A} -module structure on the same space. In other words,

$$(6.4) \quad \Phi_{\text{inv}} \circ \{\ker(\varsigma)\} = \{0\},$$

and therefore it is natural to consider Φ_{inv} as a right \mathcal{A} -module.

Furthermore, let us assume that the space \mathcal{L} admits a complement \mathcal{L}^\perp in Φ_{inv} , which is invariant under the action \varkappa , and which is a submodule of Φ_{inv} . Without

a lack of generality, we may assume that \mathcal{L}^\perp is also $*$ -invariant. We see that \mathcal{L}^\perp is naturally isomorphic to Γ_{inv} . Let $p_{\mathcal{L}}: \Phi_{inv} \rightarrow \mathcal{L}$ be the projection map associated to the spitting $\Phi_{inv} = \mathcal{L} \oplus \mathcal{L}^\perp$.

The space \mathcal{L} , equipped with the induced maps $\{*, \varkappa, \circ\}$ determines a bicovariant $*$ -bimodule $\Psi \leftrightarrow \mathcal{A} \otimes \mathcal{L}$ over G , with the associated braiding τ .

Finally, let us assume that the implication

$$(6.5) \quad \varsigma(q) = a \in \mathcal{R} \Rightarrow \pi_{\mathcal{L}}(q^{(2)})\pi_{\mathcal{L}}[\kappa'(q^{(1)})q^{(3)}] = 0$$

holds in the corresponding τ -exterior algebra \mathcal{L}^\wedge . Here $\pi_{\mathcal{L}} = p_{\mathcal{L}}\pi'$. The above condition is equivalent to a possibility of introducing consistently a ‘partial transposed commutator’ map $\Delta: \Gamma_{inv} \rightarrow \mathcal{L} \otimes \mathcal{L}/\text{im}(I + \tau)$ by the formula

$$(6.6) \quad \Delta\pi(a) = \pi_{\mathcal{L}}(q^{(2)})\pi_{\mathcal{L}}[\kappa'(q^{(1)})q^{(3)}], \quad \varsigma(q) = a.$$

Let us consider a $*$ -algebra \mathcal{B} , equipped with a free action $E: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}'$ of H . Let $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ be the $*$ -homomorphism given by

$$F = (\text{id} \otimes \varsigma)E.$$

Let $\mathcal{V} \subseteq \mathcal{B}$ be the F -fixed point subalgebra, and $i: \mathcal{V} \rightarrow \mathcal{B}$ the inclusion map. By definition, $P = (\mathcal{B}, i, F)$ is a quantum principal G -bundle over a quantum space M described by \mathcal{V} .

Let \mathfrak{hor}_P be the horizontal algebra, associated to P and \mathcal{L} , as described at the beginning of the paper. Let $\nabla: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ be a linear map defined by

$$(6.7) \quad \nabla(b \otimes \vartheta) = \sum_k b_k \otimes [\pi_{\mathcal{L}}(q_k)]\vartheta,$$

where $E(b) = \sum_k b_k \otimes q_k$.

Lemma 6. *The map ∇ is a hermitian first-order antiderivation on \mathfrak{hor}_P , intertwining the map F^\wedge . Moreover, there exists the curvature map associated to ∇ .*

Proof. From the definition (6.7) it follows that ∇ acts on \mathcal{B} as a derivation. A direct computation gives

$$\begin{aligned} \nabla(\vartheta b) &= \sum_k \nabla(b_k)(\vartheta \circ \varsigma(q_k)) = \sum_k (b_k \otimes \pi_{\mathcal{L}}(q_k^{(1)}))(\vartheta \circ \varsigma(q_k^{(2)})) \\ &= (-1)^{\partial\vartheta} \sum_k b_k \otimes [(\vartheta \circ \varsigma(q_k^{(1)}))\pi_{\mathcal{L}}(q_k^{(2)})] = (-1)^{\partial\vartheta} \vartheta \nabla(b), \end{aligned}$$

and we conclude that ∇ satisfies the graded Leibniz rule. We also see that ∇ commutes with the action F^\wedge . Furthermore,

$$\begin{aligned} \nabla[(b \otimes \vartheta)^*] &= \sum_k \nabla[b_k^* \otimes (\vartheta^* \circ \varsigma(q_k^*))] = \sum_k b_k^* \otimes [\pi_{\mathcal{L}}(q_k^{(1)*})(\vartheta^* \circ \varsigma(q_k^{(2)*}))] \\ &= \sum_k (-1)^{\partial\vartheta} \vartheta^* [b_k^* \otimes \pi_{\mathcal{L}}(q_k^*)] = [\nabla(b \otimes \vartheta)]^*, \end{aligned}$$

which proves the hermicity of ∇ .

Let us prove that there exists the curvature map. We compute

$$\nabla^2(b) = \sum_k b_k \pi_{\mathcal{L}}(q_k^{(1)})\pi_{\mathcal{L}}(q_k^{(2)}),$$

and hence if the curvature map exists, it is necessarily given by the formula

$$\varrho_{\nabla}(q) = -\pi_{\mathcal{L}}(q^{(1)})\pi_{\mathcal{L}}(q^{(2)}).$$

We have to check the consistency of the above formula. However, this is a direct consequence of the self-consistency of the partial commutator map Δ , given by (6.6). Explicitly, two maps are related by

$$(6.8) \quad \varrho_D(\vartheta) = -\frac{1}{2}\Delta(\vartheta),$$

where now $\vartheta \in \Gamma_{inv}$. Indeed, we have

$$(6.9) \quad \tau(\pi_{\mathcal{L}}(q^{(1)}) \otimes \pi_{\mathcal{L}}(q^{(2)})) = \pi_{\mathcal{L}}(q^{(1)}) \otimes \pi_{\mathcal{L}}(q^{(2)}) - \pi_{\mathcal{L}}(q^{(2)}) \otimes \pi_{\mathcal{L}}(\kappa'(q^{(1)})q^{(3)}),$$

and (6.8) follows by projecting down to \mathcal{L}^\wedge . It is worth observing that (6.5) and (6.8) prove also the automatical compatibility between the calculus Γ and the curvature ϱ_{∇} . \square

Let us fix a basis $\{\theta_1, \dots, \theta_n\}$ in the space \mathcal{L} . Let us consider the elements $c_1, \dots, c_n \in \mathcal{K}$ satisfying

$$\text{ad}_*(c_i) = \sum_j c_j \otimes u_{ji} \quad \pi'(c_i) = \theta_i,$$

where u_{ji} are the matrix elements of $\kappa: \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{A}$. Let us define the elements $b_{\alpha i} \in \mathcal{B}$ and $f_\alpha \in \mathcal{V}$ by equalities

$$\sum_\alpha b_{\alpha i} E(f_\alpha) = 1 \otimes c_i \quad F(b_{\alpha i}) = \sum_j b_{\alpha j} \otimes u_{ji}.$$

A direct calculation shows that (2.3) holds, and hence

Proposition 7. *The map ∇ determines a frame structure on a quantum principal bundle P .* \square

As a concrete class of examples of the presented construction, let us mention standard quantum homogeneous spaces (equipped with the appropriate differential structure Φ). In this case $\mathcal{B} = \mathcal{A}'$ and $E = \phi'$. Another generic class of examples is based on quantum classifying spaces [8], where we take \mathcal{B} to be the universal $*$ -algebra generated by elements ψ_{ki} , where $k \in \{1, \dots, d\}$ and $i \in \{1, \dots, n\}$, modulo the relations $\sum_k \psi_{ki}^* \psi_{kj} = \delta_{ij} \cdot 1$. The base space M is one of the classifying spaces for G , and P is the analog of the universal G -bundle. Differential calculus constructed using the presented methods is particularly suitable [5] for constructing examples of quantum characteristic classes.

6.2. Classical Structure Groups

A specially interesting class of examples is given by quantum frame bundles with classical [6] structure groups G . In this context we can further assume that the right \mathcal{A} -module structure \circ on \mathbb{V} is trivial, and that the frame structure ∇ is compatible with the classical differential calculus on G . This is equivalent to

$$(6.10) \quad \vartheta \circ a = \epsilon(a)\vartheta$$

$$(6.11) \quad \varrho_{\nabla}(ab) = \epsilon(b)\varrho_{\nabla}(a) + \epsilon(a)\varrho_{\nabla}(b)$$

respectively, and in particular all flip-over operators become the standard transpositions.

If we consider $G = \mathrm{SO}(n)$ and assume that u is the standard representation of G in $\mathbb{V} = \mathbb{C}^n$, then if $n = 2k + 1$ the unique solution for the right-module structure is given by (6.10). On the other hand, in the case $n = 2k$ we have also a possible solution

$$(6.12) \quad \vartheta \circ u_{ij} = -\delta_{ij} \vartheta.$$

For $k \geq 2$ this will be the unique nontrivial solution for the right-module structure. We see that $-\tau$ is the standard transposition on \mathbb{V} and hence \mathbb{V}^\wedge is consisting of the symmetric tensors over \mathbb{V} . Frame structures based on such $\{\mathbb{V}, \circ\}$ give interesting variants of the ‘fermionic’ differential calculi on the base space.

The presented formalism effectively incorporates [7] the basic elements of the classical theory of Kähler manifolds. In this context the structure group is reduced to $G = \mathrm{U}(k)$.

However, even if \circ is trivial and ∇ is arbitrary the minimal calculus Γ will be non-standard.

In what follows we shall analyze in more details framed complex line bundles. This is specified by $G = \mathrm{SO}(2)$, and the standard 2-dimensional representation

$$\varkappa = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$$

where φ is the angle function. Let us assume that the $*$ -structure is trivial. We have to fix the \circ -operation in the space $\mathbb{V} = \mathbb{C}^2$. The solutions are given by

$$\begin{aligned} \psi \circ u &= \lambda \psi & \psi^* &= \theta_1 + i\theta_2 \\ \psi^* \circ u &= \lambda \psi^* & \psi &= \theta_1 - i\theta_2 \end{aligned}$$

where $0 \neq \lambda \in \mathfrak{R}$. In the basis $\{\psi, \psi^*\}$, the flip-over operator τ is given by

$$\tau = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

and relations in the exterior algebra \mathbb{V}^\wedge are given by

$$\lambda \psi \psi^* = -\psi^* \psi \quad \psi^2 = \psi^{*2} = 0,$$

where we have assumed that $\lambda \neq -1$.

Let us consider a quantum principal G -bundle P , equipped with a frame structure ∇ . It turns out that the associated calculus on G will be generally infinite-dimensional (this is caused by the fact that the algebra \mathcal{B} may be very complicated, and the components of the curvature can be in principle arbitrarily complicated elements of this algebra). However, in some special cases we can perform a further reduction. For example, let us assume that ∇ is such that its curvature takes values from $\mathbb{V}^\wedge \subseteq \mathfrak{hor}_P$. In other words,

$$\varrho_\nabla(a) = c(a) \psi \psi^*$$

with $c: \mathcal{A} \rightarrow \mathbb{C}$. If ϱ_∇ is non-vanishing then the minimal calculus Γ compatible with the frame structure ∇ will be 1-dimensional, and

$$\vartheta \circ u = \lambda^2 \vartheta$$

for each $\vartheta \in \Gamma_{inv}$. This calculus is based on the ideal

$$\mathcal{R} = \mathrm{gen}\{1 + \lambda^2 - u - \lambda^2 u^{-1}\}.$$

For the end of this subsection, let us illustrate the main structural elements of the presented theory in the context of quantum line bundles. We shall assume that the base space M is a classical compact smooth manifold. However, many of the formulas hold in the full generality.

Let Λ be a complex line bundle over M , and let \mathcal{F} be the space of smooth sections of Λ . Let γ be a given automorphism of the algebra $\mathcal{V} = S(M)$. We shall assume that \mathcal{F} is equipped with a \mathcal{V} bimodule structure consisting of the standard multiplication on the left, and the right \mathcal{V} -module structure specified by

$$\xi f = \gamma(f)\xi.$$

Now, we can apply the reconstruction procedure as described in [9], and construct a quantum principal G -bundle P over M , starting from $\{\mathcal{F}, \gamma\}$. The algebra \mathcal{B} is given by

$$\mathcal{B} = \sum_{n \in \mathbb{Z}}^{\oplus} \mathcal{F}_n$$

where $\mathcal{F}_k = \mathcal{F}^{\otimes k}$ and $\mathcal{F}_{-k} = \bar{\mathcal{F}}^{\otimes k}$, for $k \in \mathbb{N}$. The tensor product is taken over \mathcal{V} , and $\bar{\mathcal{F}}$ is the conjugate bimodule. The above definition is completed by saying $\mathcal{F}_0 = \mathcal{V}$.

The frame structure is determined by a single map $X: \mathcal{B} \rightarrow \mathcal{B}$, via the formula

$$(6.13) \quad \nabla(b) = X(b) \otimes \psi + \lambda X^* L(b) \otimes \psi^*,$$

where $L: \mathcal{B} \rightarrow \mathcal{B}$ is an automorphism given by $(L|_{\mathcal{F}_n}) = \lambda^n$. So that we have

$$\psi b = L(b)\psi \quad L(b^*) = L^{-1}(b)^*$$

for each $b \in \mathcal{B}$. The map X should satisfy the completeness axiom, and the following conditions

$$(6.14) \quad XX^* = X^*X \quad |_{\mathcal{V}}$$

$$(6.15) \quad X(\mathcal{F}_n) \subseteq \mathcal{F}_{n-1}$$

$$(6.16) \quad X(bq) = bX(q) + X(b)L(q).$$

It follows that

$$\begin{aligned} [X, X^*](bq) &= [X, X^*](b)L(q) + L^{-1}(b)[X, X^*](q) \\ X^*(bq) &= X^*(b)q + L^{-1}(b)X^*(q). \end{aligned}$$

The corresponding frame structure ∇ is always regular. The curvature map is given by

$$(6.17) \quad \varrho_{\nabla}(u^n) = \sum_{\alpha} q_{\alpha} [X, X^*] L(b_{\alpha}) \otimes \psi^* \psi,$$

where $n \in \mathbb{Z}$ and the elements $q_{\alpha} \in \mathcal{F}_{-n}$ and $b_{\alpha} \in \mathcal{F}_n$ satisfy

$$\sum_{\alpha} q_{\alpha} b_{\alpha} = 1.$$

A direct computation gives

$$(6.18) \quad \varrho_{\nabla}(u^n) = \left\{ \sum_{k=0}^{n-1} \lambda^{2k} \gamma^{-k} \right\} \varrho_{\nabla}(u)$$

$$(6.19) \quad \varrho_{\nabla}(u^{-n}) = -\lambda^{-2} \gamma \left\{ \sum_{k=0}^{n-1} \lambda^{-2k} \gamma^k \right\} \varrho_{\nabla}(u),$$

for each $n \in \mathbb{N}$. In the above formulas γ is trivially extended to $\Omega_M^2 = \mathcal{V} \otimes \{\psi^* \psi\}$.

Therefore, the minimal calculus Γ compatible with a single operator ∇ will be generally higher-dimensional, including the universal calculus as a possible minimal solution.

It is worth noticing that the induced calculus over M is non-classical. Indeed, we have the following natural decomposition

$$\Omega_M^1 = \Omega_M^+ \oplus \Omega_M^-,$$

where $\Omega_M^+ = \mathcal{F}_{-1} \otimes \{\psi\}$ and $\Omega_M^- = \mathcal{F}_1 \otimes \{\psi^*\}$, are the spaces playing the role of differential forms of holomorphic and antiholomorphic types. Therefore in Ω_M we have non-trivial commutation relations.

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